Hansheng Wang

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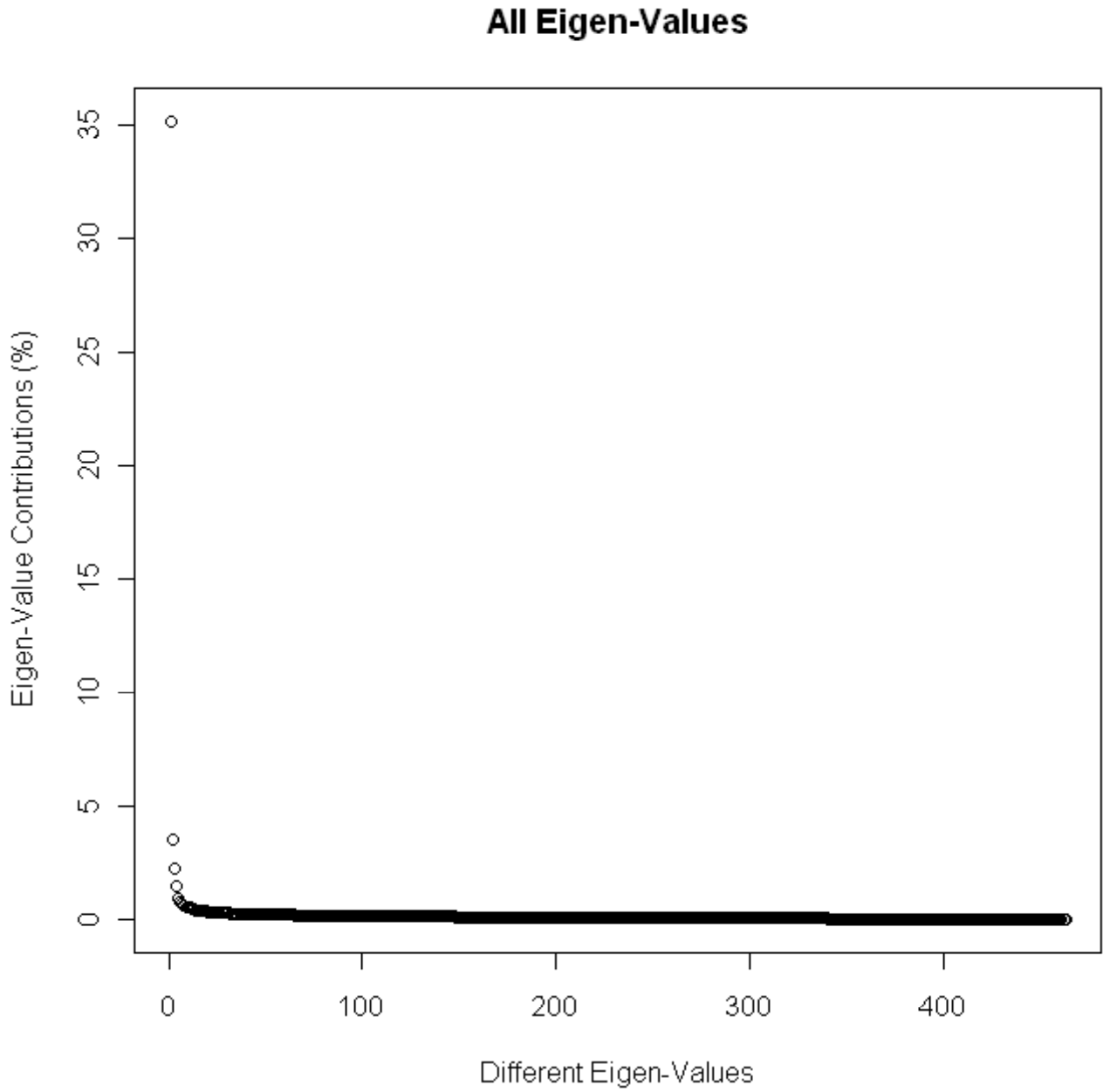
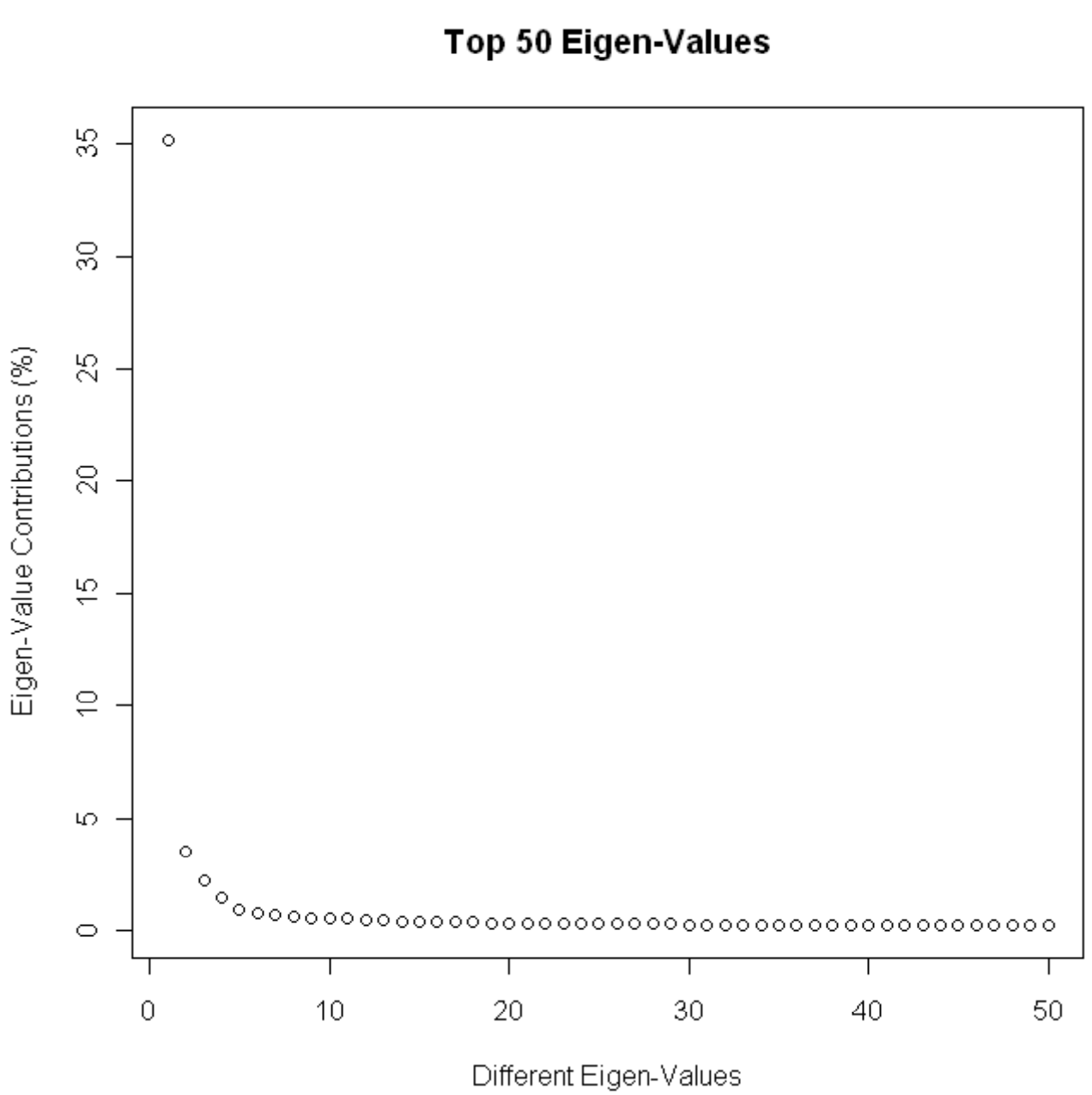
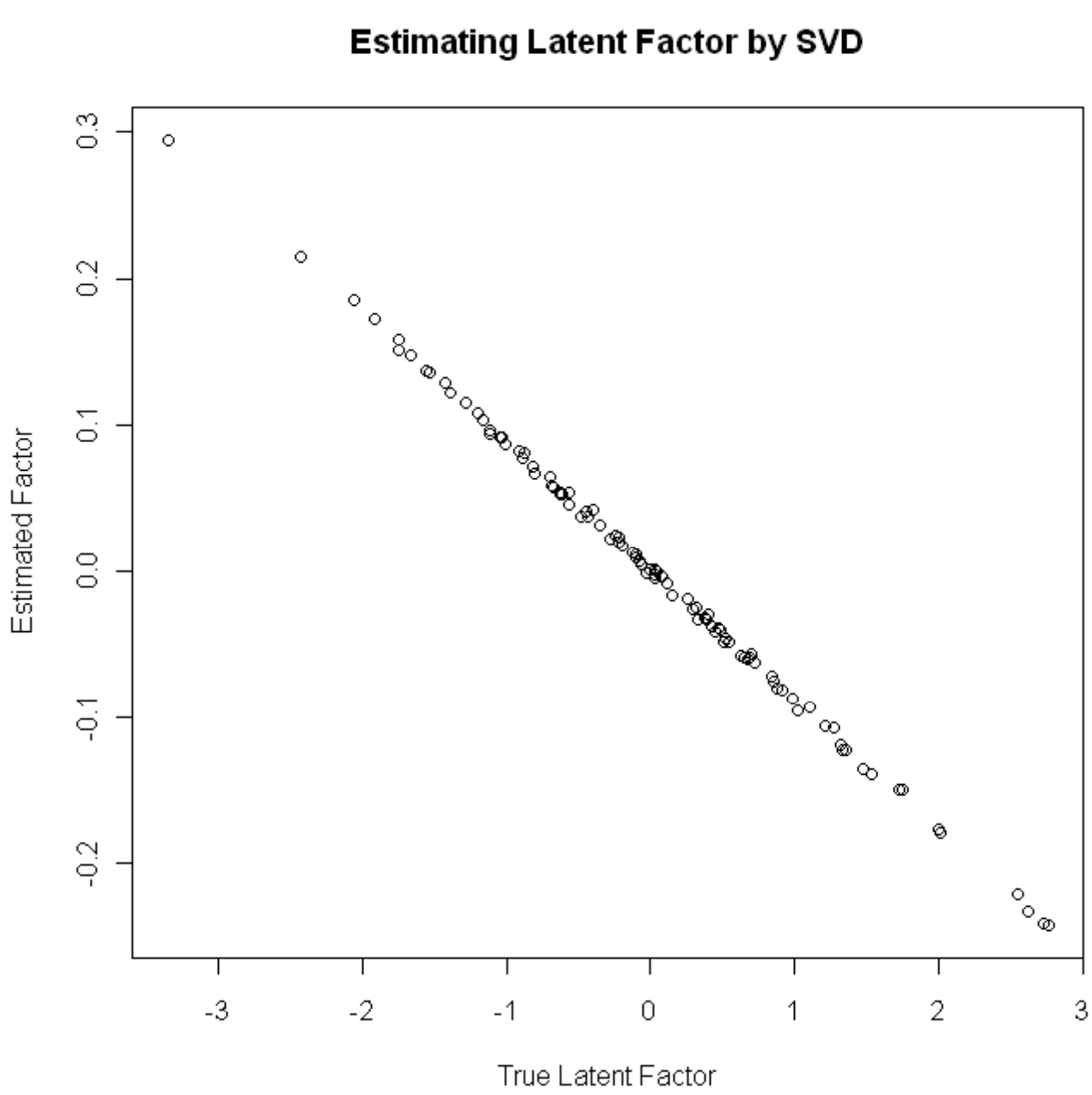

All Eigen-Values
Eigen-Value Contributions (%)
0
5
10
15
20
25
30
35
0
100
200
300
400
Different Eigen-Values


Top 50 Eigen-Values
Eigen-Value Contributions (%)
0
5
10
15
20
25
30
35
0
10
20
30
40
50
Different Eigen-Values


Estimating Latent Factor by SVD
Estimated Factor
True Latent Factor
0.3
0.2
0.1
0.0
-0.1
-0.2
-3
-2
-1
0
1
2
3

- To model the regression relationship between Y_i and X_i, we assume that

$$Y_i = X_i^\top \theta + \varepsilon_i, \tag{2.1}$$

where ε_i is a random noise with mean 0 and variance σ_ε^2; $\theta = (\theta_1, \cdots, \theta_p)^\top \in \mathbb{R}^p$ is a p-dimensional coefficient vector and its true value is given by $\theta_0 = (\theta_{01}, \cdots, \theta_{0p})^\top \in \mathbb{R}^p$.

- To model the factor structure, we follow Fan et al. (2008) and assume

$$X_i = BZ_i + \widetilde{X}_i, \tag{2.2}$$

where $Z_i = (Z_{i1}, \cdots, Z_{id})^\top \in \mathbb{R}^d$ is a d-dimensional latent factor, $B = (b_{jk}) \in \mathbb{R}^{p \times d}$ is the loading matrix, and $\widetilde{X}_i = (\widetilde{X}_{i1}, \cdots, \widetilde{X}_{ip})^\top \in \mathbb{R}^p$ represents the information contained in X_i but missed by Z_i.

reflect the endogeneity problem, we allow that ε_i to be correlated with X_i and Y_i through the common factor Z_i as

$$\varepsilon_i = Z_i^\top \alpha + \tilde{\varepsilon}_i, \tag{2.3}$$

where $\alpha = (\alpha_1, \cdots, \alpha_d)^\top \in \mathbb{R}^d$ is a d-dimensional vector and its true value is given by $\alpha_0 \in \mathbb{R}^d$. Moreover, $\tilde{\varepsilon}_i$ is some random noise independent of both Z_i and $\widetilde{X}_i$. We then should have $\text{var}(\tilde{\varepsilon}_i) = \tilde{\sigma}_\varepsilon^2 \leq \text{var}(Y_i) = 1$.

- Profiled Response: $\widetilde{Y}_i = Y_i - Z_i^\top \gamma_0$ with $\gamma_0 = B^\top \theta_0 + \alpha_0$.

- Profiled Predictor and Noise: $\widetilde{X}_i$ and $\tilde{\varepsilon}_i$.

- Profiled Regression Model: $\widetilde{Y}_i = \widetilde{X}_i^\top \theta_0 + \tilde{\varepsilon}_i$.

- Let $(\hat{\lambda}_j, \hat{V}_j)$ be the jth $(1 \leq j \leq n)$ leading eigenvalue-eigenvector pair for the matrix $\mathbb{X}\mathbb{X}^\top/(np) \in \mathbb{R}^{n \times n}$. Thus, we should have $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_n$.

- Thus, if we define an eigenvalue ratio criterion as $\hat{\lambda}_j/\hat{\lambda}_{j+1}$ with $\hat{\lambda}_0 = 1$ and $1 \leq j \leq (n-1)$, we should expect its maximum value to happen at $j = d_0$.

- Consequently, the true structure dimension can be estimated by

$$\hat{d} = \text{argmax}_{0 \leq j \leq d_{\max}} (\hat{\lambda}_j/\hat{\lambda}_{j+1}),$$

where $d_{\max}$ is a pre-specified maximum factor dimension.

Theorem 1. *Assume technical conditions (A1)–(A3) as given in the Appendix A, then we should have* $\alpha = \alpha_0$ *and* $\beta = \beta_0$.

With a correctly specified factor dimension (i.e., $d = d_0$), we can subsequently construct a least squares type objective function as

$$\mathcal{O}(\mathbb{Z}, B) = (np)^{-1} \sum_{j=1}^{p} \left\| \mathbb{X}_j - \mathbb{Z}\beta_j \right\|^2$$

with $\beta_j = (b_{j1}, \cdots, b_{jd})^\top \in \mathbb{R}^d$. We know immediately that $B = (\beta_1, \cdots, \beta_p)^\top \in \mathbb{R}^{p \times d}$. Then, $\mathcal{S}(\mathbb{Z})$ can be estimated by minimizing $\mathcal{O}(\mathbb{Z}, B)$ with respect to both $\mathbb{Z} \in \mathbb{R}^{n \times d}$ and $B \in \mathbb{R}^{p \times d}$.

To quantify the estimation accuracy of $\mathcal{S}(\widehat{\mathbb{Z}})$, the following two discrepancy measures are considered. They are, respectively,

$$D_1(\mathbb{Z}, \widehat{\mathbb{Z}}) = n^{-1} tr\left\{\mathbb{Z}^\top Q(\widehat{\mathbb{Z}})\mathbb{Z}\right\} \text{ and } D_2(\mathbb{Z}, \widehat{\mathbb{Z}}) = tr\left\{H(\mathbb{Z}) - H(\widehat{\mathbb{Z}})\right\}^2.$$

Theorem 2. *Assume $d = d_0$ and the technical conditions (A1)–(A3) as given in the Appendix A, then we should have both $D_1(\mathbb{Z}, \widehat{\mathbb{Z}}) = O_p(n^{-1})$ and $D_2(\mathbb{Z}, \widehat{\mathbb{Z}}) = O_p(n^{-1})$.*

With an estimated dimension d_0 of $\mathcal{S}(Z)$, we can get rough predicted data as $\widehat{\mathbb{Y}} = Q(\widehat{\mathbb{Z}})\mathbb{Y}$ and $\widehat{\mathbb{X}} = Q(\widehat{\mathbb{Z}})\mathbb{X}$, with $\widehat{\mathbb{X}} = (\widehat{\mathbb{X}}_1, \cdots, \widehat{\mathbb{X}}_p) \in \mathbb{R}^{n \times p}$.

- Subsequently, the simple method of SIS can be applied to $\widehat{\mathbb{Y}}$ and $\widehat{\mathbb{X}}$ directly, and the resulting estimate is path consistent (Leng et al., 2006). We refer to such a method as PIS.

- More specifically, PIS estimates θ_j by $\hat{\theta}_j = (n^{-1}\widehat{\mathbb{X}}_j^\top \widehat{\mathbb{X}}_j)^{-1}(n^{-1}\widehat{\mathbb{Y}}^\top \widehat{\mathbb{X}}_j)$.

Theorem 3. *Assume $d = d_0$ and the technical conditions (A1)–(A3) as given in the*

Previous subsection proves that PIS is path consistent, which implies that $P(\mathcal{M}_T = \mathcal{M}_{(|\mathcal{M}_T|)}) \to 1$ as $n \to \infty$. However, for a real application, the value of $|\mathcal{M}_T|$ is unknown. Thus, even if the solution path is given, one still needs a statistically sound criterion to decide which model in $\mathbb{M}$ is mostly plausible. To this end, we proposed here the following heuristic BIC-type selection criterion,

$$\mathrm{BIC}(\mathcal{M}) = \log \mathrm{RSS}(\mathcal{M}) + |\mathcal{M}| \cdot \log n \cdot (\log p/n), \tag{3.1}$$

where $\mathrm{RSS}(\mathcal{M}) = \|\widehat{\mathbb{Y}} - \sum_{j\in\mathcal{M}} \hat{\theta}_j \widehat{\mathbb{X}}_j\|^2$ is the residual sum of squares. Then the best model can be selected as $\widehat{\mathcal{M}} = \mathrm{argmin}_{\mathcal{M}\in\mathbb{M}} \mathrm{BIC}(\mathcal{M})$.

Step (1) (*Initialization*). Set $\mathcal{M}^*_{(0)} = \emptyset$ and $\widehat{\mathbb{Y}}^{(0)} = \widehat{\mathbb{Y}}$, i.e., the factor profiled response.

Step (2) (*Sequential Screening*).

(2.1) (*Estimation*). In the kth step ($k \geq 1$), we are given $\mathcal{M}^*_{(k-1)}$ and also $\widehat{\mathbb{Y}}^{(k-1)}$. Then, for every $j \in \mathcal{M}_F \backslash \mathcal{M}^*_{(k-1)}$, estimate its regression coefficient as $\hat{\theta}_j^{(k)} = \{\widehat{\mathbb{Y}}^{(k-1)\top}\widehat{\mathbb{X}}_j\}/\|\widehat{\mathbb{X}}_j\|^2$ and its correlation coefficient with the response as $\hat{\zeta}_j^{(k)} = \{\widehat{\mathbb{Y}}^{(k-1)\top}\widehat{\mathbb{X}}_j\}/\{\|\widehat{\mathbb{Y}}^{(k-1)}\| \cdot \|\widehat{\mathbb{X}}_j\|\}$.

(2.2) (*Screening*). We then find $a_k = \operatorname{argmax}_{j \in \mathcal{M}_F \backslash \mathcal{S}^{(k-1)}} |\hat{\zeta}_j^{(k)}|$ and update $\mathcal{M}^*_{(k)} = \mathcal{M}^*_{(k-1)} \bigcup \{a_k\}$ accordingly.

(2.3) (*Elimination*). According to a_k, we then get an updated response vector as $\widehat{\mathbb{Y}}^{(k)} = \widehat{\mathbb{Y}}^{(k-1)} - \hat{\theta}_j^{(k)}\widehat{\mathbb{X}}_j$ with $j = a_k$.

Step (3) (*Solution Path*). Iterating Step (2) for a total of n times, which leads a total of $n+1$ nested candidate models. We then collect those models by a solution path $\mathbb{M}^* = \{\mathcal{M}^*_{(k)} : 0 \leq k \leq n\}$ with $\mathcal{M}^*_{(k)} = \{a_1, \cdots, a_k\}$ for $k > 0$.

Step (4) (*Model Selection*). Select the best model as $\widehat{\mathcal{M}}^* = \operatorname{argmin}_{\mathcal{M}^*_{(k)} \in \mathbb{M}^*} \mathrm{BIC}(\mathcal{M}^*_{(k)})$.

Example 1. This is an example borrowed from Fan and Lv (2008). Specifically, we fix $d_0 = 1$, $p = 5000$, and $n = 150$. Z_i is generated from $N(0, 1)$. X_i is then simulated as (2.2), where $b_{jk} = 1$ and $\widetilde{X}_i$ follows a p-dimensional standard normal distribution. Following Fan and Lv (2008), we assume the first $|\mathcal{M}_T| = 3$ predictors to be relevant and their coefficients are given by $\theta_{0j} = 5$ for $1 \leq j \leq |\mathcal{M}_T|$. Accordingly, $\theta_{0j} = 0$ for every $j > |\mathcal{M}_T|$. Subsequently, Y_i is given by (2.1), where ε_i follows (2.3) with $\alpha_0 = 0.8\sigma_\varepsilon$ and $\tilde{\sigma}_\varepsilon = 0.6\sigma_\varepsilon$. Lastly, σ_ε^2 is particularly selected so that the signal-to-noise ratio, i.e., SNR=$\text{var}(X_i^\top \theta_0)/\sigma_\varepsilon^2$, is given by 1, 2, or 5.

Signal Noise Ratio	Variable Selection Method	% of Correct Zeros	% of Incorrect Zeros	% of Correct fit	Average Model Size	Absolute Estimation Error
						EXAMPLE 1
1	SIS	100.0	77.2	0.0	1.0	25.4
	PIS	100.0	95.8	0.5	0.1	14.6
	PSS	100.0	95.8	0.5	0.1	14.6
2	SIS	100.0	70.3	0.0	1.0	21.3
	PIS	100.0	46.3	40.0	1.6	7.9
	PSS	100.0	43.3	45.5	1.7	7.4
5	SIS	100.0	67.0	0.0	1.0	18.4
	PIS	100.0	0.2	99.5	3.0	1.0
	PSS	100.0	0.0	100.0	3.0	0.9

As our first step, we need to estimate the dimension of the latent factor. We find that the first eigenvalue of the matrix $\mathbb{X}\mathbb{X}^\top/(np)$ is as large as $\hat{\lambda}_1 = 35.4\%$ while the second one is as small as $\hat{\lambda}_2 = 3.5\%$. The big difference as demonstrated between $\hat{\lambda}_1$ and $\hat{\lambda}_2$ suggests that the true factor dimension might be $d_0 = 1$. Such a conjecture is formally confirmed by MERC. We then fix $d = 1$ throughout the rest of this example. Thereafter, the factor subspace $\mathcal{S}(\widehat{\mathbb{Z}})$ can be estimated and the profiled data $(\widehat{\mathbb{Y}}, \widehat{\mathbb{X}})$ can be produced.

For a real problem like this, the value of θ_0 is unknown. We thus have to rely on out-of-sample testing to compare different methods' estimation and/or prediction accuracy. We then conducted a total of 200 random experiments. For each experiment, we randomly split the entire dataset $\mathcal{D} = \{1, \cdots, 464\}$ into two parts. That is $\mathcal{D} = \mathcal{D}_0 \bigcup \mathcal{D}_1$ with $|\mathcal{D}_0| = n_0 = 400$ as the training data and $|\mathcal{D}_1| = n_1 = 64$ as the testing data. Accordingly, we write $\mathbb{X}_0 = \{X_i : i \in \mathcal{D}_0\} \in \mathbb{R}^{n_0 \times p}$, $\mathbb{Y}_0 = \{Y_i : i \in \mathcal{D}_0\} \in \mathbb{R}^{n_0}$, $\mathbb{X}_1 = \{X_i : i \in \mathcal{D}_1\} \in \mathbb{R}^{n_1 \times p}$, and $\mathbb{Y}_1 = \{Y_i : i \in \mathcal{D}_1\} \in \mathbb{R}^{n_1}$. Notations for $(\widehat{\mathbb{X}}_0, \widehat{\mathbb{X}}_1)$, $(\widehat{\mathbb{Y}}_0, \widehat{\mathbb{Y}}_1)$, and $(\widehat{\mathbb{Z}}_0, \widehat{\mathbb{Z}}_1)$ are defined accordingly.

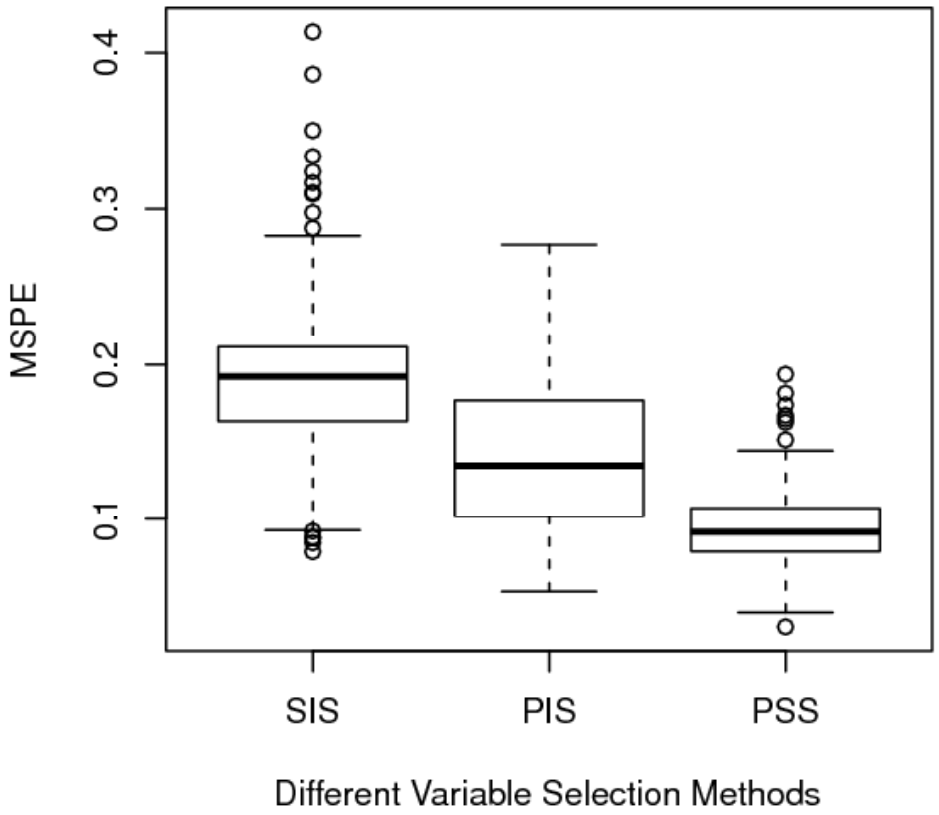


Figure 1: The real supermarket example. Boxplots for the median squared prediction errors (MSPE) based on 200 random replications.