

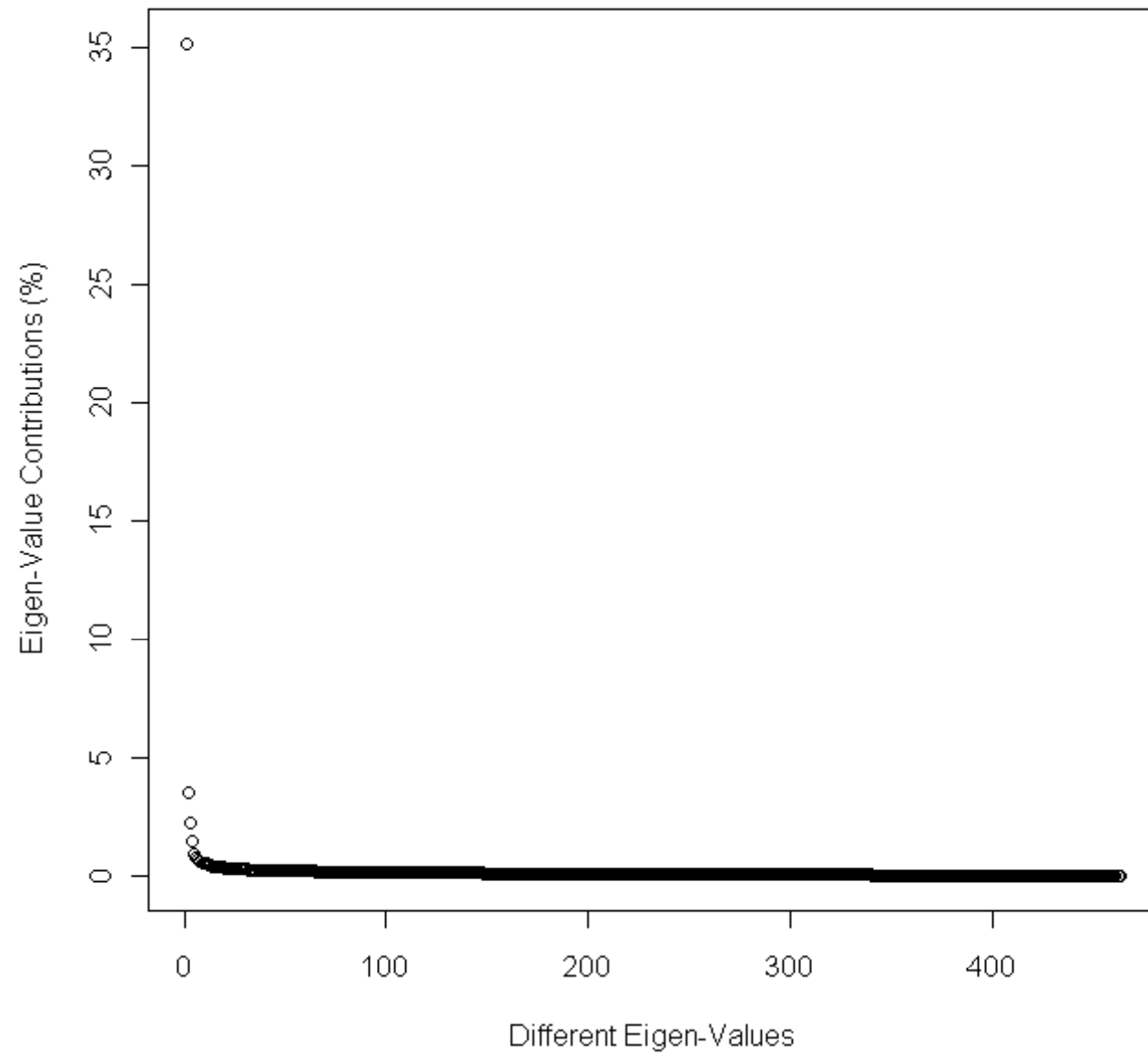
Hansheng Wang



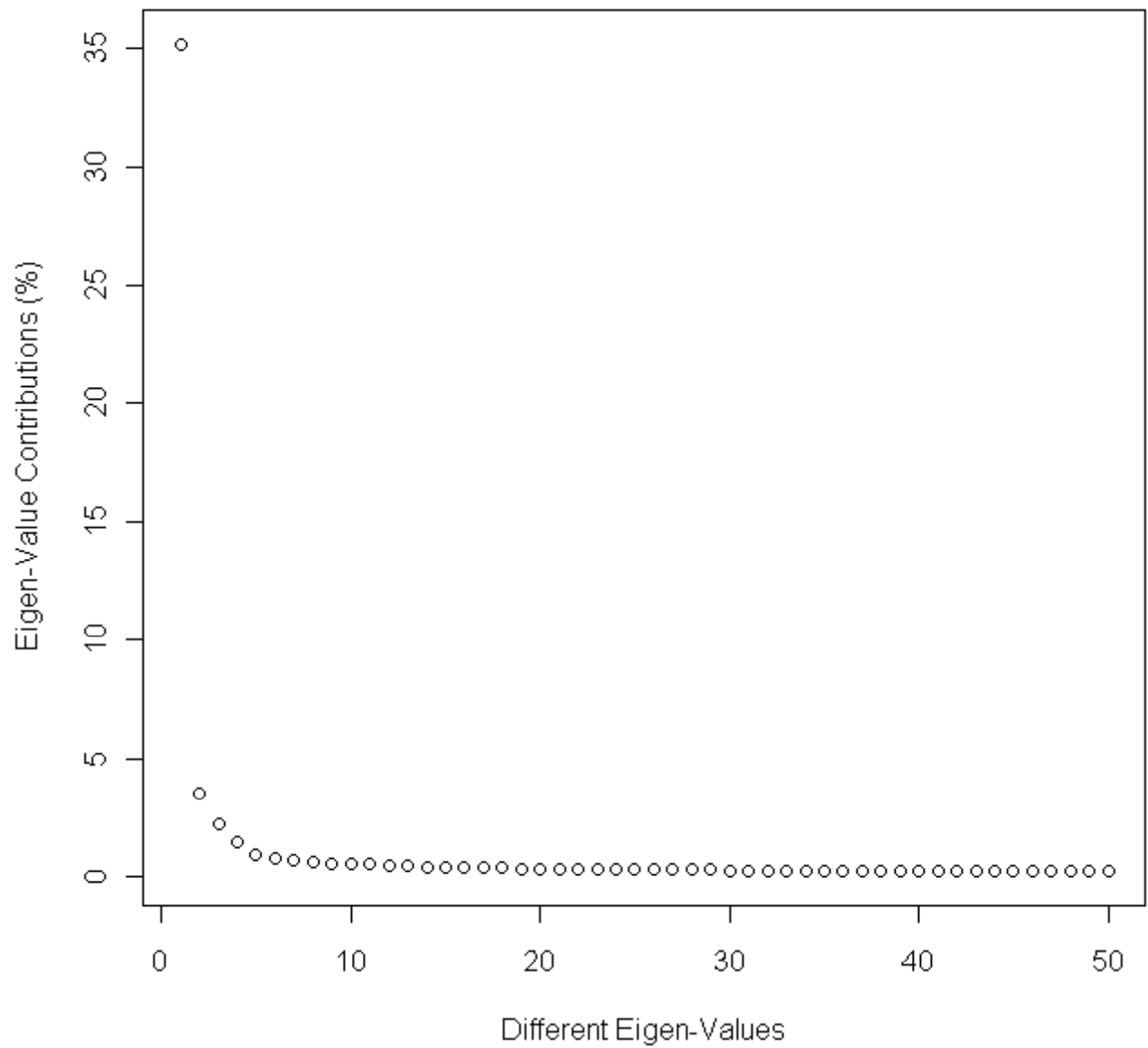




All Eigen-Values

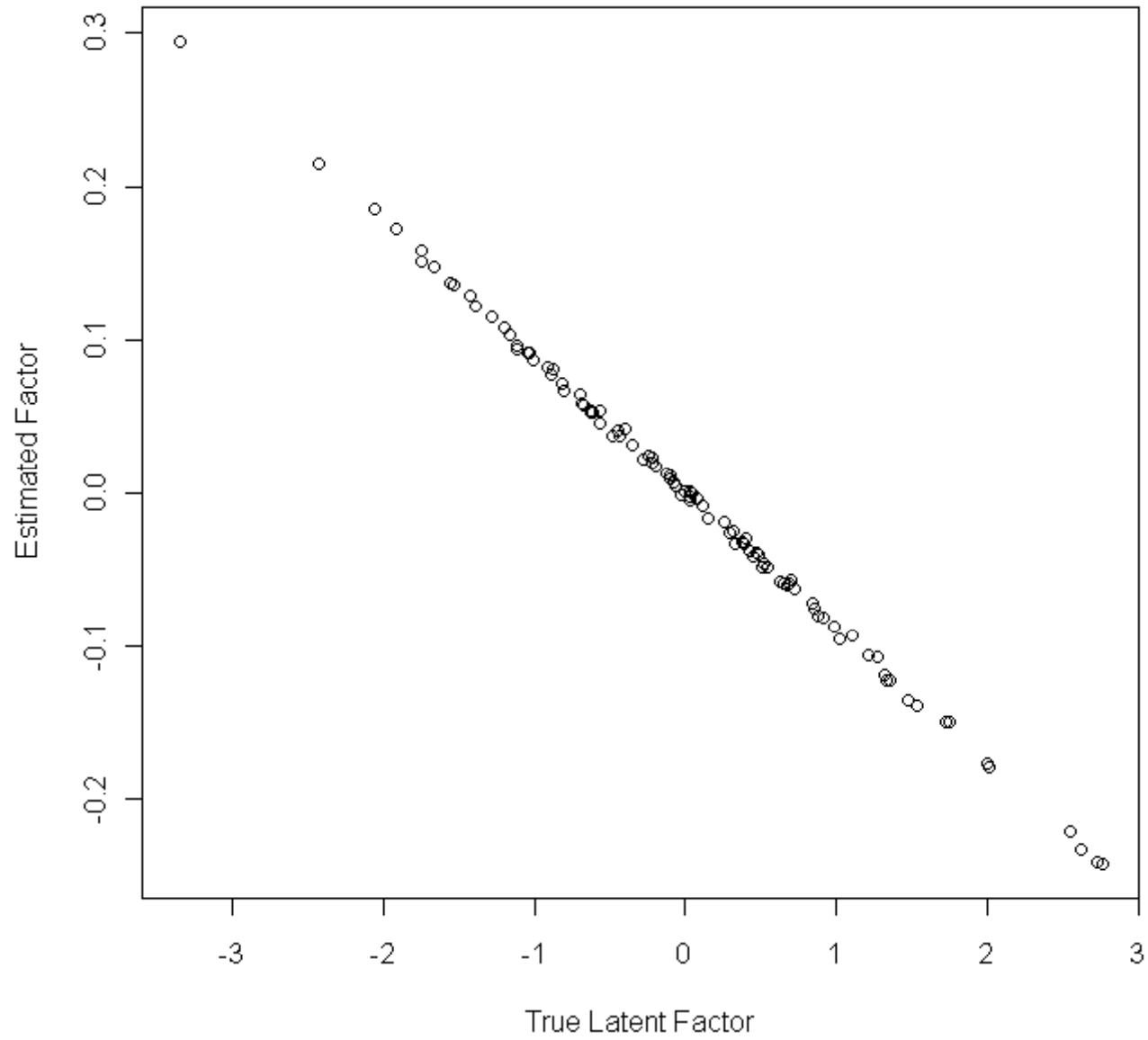


Top 50 Eigen-Values





Estimating Latent Factor by SVD



- To model the regression relationship between Y_i and X_i , we assume that

$$Y_i = X_i^\top \theta + \varepsilon_i, \quad (2.1)$$

where ε_i is a random noise with mean 0 and variance σ_ε^2 ; $\theta = (\theta_1, \dots, \theta_p)^\top \in \mathbb{R}^p$ is a p -dimensional coefficient vector and its true value is given by $\theta_0 = (\theta_{01}, \dots, \theta_{0p})^\top \in \mathbb{R}^p$.

- To model the factor structure, we follow Fan et al. (2008) and assume

$$X_i = BZ_i + \tilde{X}_i, \quad (2.2)$$

where $Z_i = (Z_{i1}, \dots, Z_{id})^\top \in \mathbb{R}^d$ is a d -dimensional latent factor, $B = (b_{jk}) \in \mathbb{R}^{p \times d}$ is the loading matrix, and $\tilde{X}_i = (\tilde{X}_{i1}, \dots, \tilde{X}_{ip})^\top \in \mathbb{R}^p$ represents the information contained in X_i but missed by Z_i .

To reflect the heterogeneity problem, we allow that ε_i to be correlated with X_i through the common factor Z_i as

$$\varepsilon_i = Z_i^\top \alpha + \tilde{\varepsilon}_i, \quad (2.3)$$

where $\alpha = (\alpha_1, \dots, \alpha_d)^\top \in \mathbb{R}^d$ is a d -dimensional vector and its true value is given by $\alpha_0 \in \mathbb{R}^d$. Moreover, $\tilde{\varepsilon}_i$ is some random noise independent of both Z_i and \tilde{X}_i . We then should have $\text{var}(\tilde{\varepsilon}_i) = \tilde{\sigma}_\varepsilon^2 \leq \text{var}(Y_i) = 1$.

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- Profiled Response: $\tilde{Y}_i = Y_i - Z_i^\top \gamma_0$ with $\gamma_0 = B^\top \theta_0 + \alpha_0$.
- Profiled Predictor and Noise: \tilde{X}_i and $\tilde{\varepsilon}_i$.
- Profiled Regression Model: $\tilde{Y}_i = \tilde{X}_i^\top \theta_0 + \tilde{\varepsilon}_i$.

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- Let $(\hat{\lambda}_j, \hat{V}_j)$ be the j th ($1 \leq j \leq n$) leading eigenvalue-eigenvector pair for the matrix $\mathbb{X}\mathbb{X}^\top/(np) \in \mathbb{R}^{n \times n}$. Thus, we should have $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \dots \geq \hat{\lambda}_n$.



- Thus, if we define an eigenvalue ratio criterion as $\hat{\lambda}_j/\hat{\lambda}_{j+1}$ with $\hat{\lambda}_0 = 1$ and $1 \leq j \leq (n - 1)$, we should expect its maximum value to happen at $j = d_0$.
- Consequently, the true structure dimension can be estimated by

$$\hat{d} = \operatorname{argmax}_{0 \leq j \leq d_{\max}} (\hat{\lambda}_j/\hat{\lambda}_{j+1}),$$

where d_{\max} is a pre-specified maximum factor dimension.

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Theorem 1. *Assume technical conditions (A1)–(A2) as given in the Appendix. If $\lambda_{1n} \rightarrow 0$ as $n \rightarrow \infty$, then we should have $P(\hat{d}^m = d_0) \rightarrow 1$ as $n \rightarrow \infty$.*

with a correctly specified factor dimension (i.e., $d = d_0$), we can subsequently construct a least squares type objective function as

$$\mathcal{O}(\mathbb{Z}, B) = (np)^{-1} \sum_{j=1}^p \|\mathbb{X}_j - \mathbb{Z}\beta_j\|^2$$

with $\beta_j = (b_{j1}, \dots, b_{jd})^\top \in \mathbb{R}^d$. We know immediately that $B = (\beta_1, \dots, \beta_p)^\top \in \mathbb{R}^{p \times d}$. Then, $\mathcal{S}(\mathbb{Z})$ can be estimated by minimizing $\mathcal{O}(\mathbb{Z}, B)$ with respect to both $\mathbb{Z} \in \mathbb{R}^{n \times d}$ and $B \in \mathbb{R}^{p \times d}$.

To quantify the estimation accuracy of $\mathcal{S}(\widehat{\mathbb{Z}})$, the following two discrepancy measures are considered. They are, respectively,

$$D_1(\mathbb{Z}, \widehat{\mathbb{Z}}) = n^{-1} \text{tr} \left\{ \mathbb{Z}^\top Q(\widehat{\mathbb{Z}}) \mathbb{Z} \right\} \text{ and } D_2(\mathbb{Z}, \widehat{\mathbb{Z}}) = \text{tr} \left\{ H(\mathbb{Z}) - H(\widehat{\mathbb{Z}}) \right\}^2.$$

Theorem 2. *Assume $d = d_0$ and the technical conditions (A1)–(A3) as given in the Appendix A, then we should have both $D_1(\mathbb{Z}, \widehat{\mathbb{Z}}) = O_p(n^{-1})$ and $D_2(\mathbb{Z}, \widehat{\mathbb{Z}}) = O_p(n^{-1})$.*

With $\hat{Y} = Q(\hat{Z})\hat{\Sigma}^{-1}Y$ and $\hat{X} = Q(\hat{Z})\hat{\Sigma}^{-1}X$, we can get $\hat{Y} \in \mathbb{R}^{n \times p}$ and $\hat{X} = Q(\hat{Z})\hat{\Sigma}^{-1}X$ with $\hat{\Sigma} = (\hat{\Sigma}_1, \dots, \hat{\Sigma}_p)$.

- Subsequently, the simple method of SIS can be applied to \hat{Y} and \hat{X} directly, and the resulting estimate is path consistent (Leng et al., 2006). We refer to such a method as PIS.
- More specifically, PIS estimates θ_j by $\hat{\theta}_j = (n^{-1}\hat{X}_j^\top \hat{X}_j)^{-1}(n^{-1}\hat{Y}^\top \hat{X}_j)$.

Theorem 3. Assume $d = d_0$ and the technical conditions (A1)–(A3) as given in the

Appendix A, then we should have $\|\hat{\theta}_j - \theta_j\| \leq \sqrt{\frac{\log(p)}{n}} \max_{1 \leq j \leq p} \|\theta_j\|$.

Previous subsection proves that PIS is path consistent, which implies that $P(\mathcal{M}_T = \mathcal{M}_{(|\mathcal{M}_T|)}) \rightarrow 1$ as $n \rightarrow \infty$. However, for a real application, the value of $|\mathcal{M}_T|$ is unknown. Thus, even if the solution path is given, one still needs a statistically sound criterion to decide which model in \mathbb{M} is mostly plausible. To this end, we proposed here the following heuristic BIC-type selection criterion,

$$\text{BIC}(\mathcal{M}) = \log \text{RSS}(\mathcal{M}) + |\mathcal{M}| \cdot \log n \cdot (\log p/n), \quad (3.1)$$

where $\text{RSS}(\mathcal{M}) = \|\widehat{\mathbf{Y}} - \sum_{j \in \mathcal{M}} \hat{\theta}_j \widehat{\mathbf{X}}_j\|^2$ is the residual sum of squares. Then the best model can be selected as $\widehat{\mathcal{M}} = \text{argmin}_{\mathcal{M} \in \mathbb{M}} \text{BIC}(\mathcal{M})$.

Step (1) (*Initialization*). Set $\mathcal{M}_{(0)}^* = \emptyset$ and $\widehat{\mathbf{Y}}^{(0)} = \widehat{\mathbf{Y}}$, i.e., the factor profiled response.

Step (2) (*Sequential Screening*).

(2.1) (*Estimation*). In the k th step ($k \geq 1$), we are given $\mathcal{M}_{(k-1)}^*$ and also

$\widehat{\mathbf{Y}}^{(k-1)}$. Then, for every $j \in \mathcal{M}_F \setminus \mathcal{M}_{(k-1)}^*$, estimate its regression coefficient as $\hat{\theta}_j^{(k)} = \{\widehat{\mathbf{Y}}^{(k-1)\top} \widehat{\mathbf{X}}_j\} / \|\widehat{\mathbf{X}}_j\|^2$ and its correlation coefficient with the response as $\hat{\zeta}_j^{(k)} = \{\widehat{\mathbf{Y}}^{(k-1)\top} \widehat{\mathbf{X}}_j\} / \{\|\widehat{\mathbf{Y}}^{(k-1)}\| \cdot \|\widehat{\mathbf{X}}_j\|\}$.

(2.2) (*Screening*). We then find $a_k = \arg \max_{j \in \mathcal{M}_F \setminus \mathcal{M}_{(k-1)}^*} |\hat{\zeta}_j^{(k)}|$ and update

$$\mathcal{M}_{(k)}^* = \mathcal{M}_{(k-1)}^* \cup \{a_k\} \text{ accordingly.}$$

(2.3) (*Elimination*). According to a_k , we then get an updated response vector $\widehat{\mathbf{Y}}^{(k)} = \widehat{\mathbf{Y}} - \hat{\theta}_{a_k}^{(k)} \widehat{\mathbf{X}}_{a_k}$ with $j = a_k$ as $\widehat{\mathbf{Y}}^{(k)}$.

Iterating Step (2) for a total of n times, which leads a total Step (3) (*Solution Path*)

of $n+1$ nested candidate models. We then collect those models by a solution

path $\mathbb{M}^* = \{\mathcal{M}_{(k)}^* : 0 \leq k \leq n\}$ with $\mathcal{M}_{(k)}^* = \{a_1, \dots, a_k\}$ for $k > 0$.

Step (4) (*Model Selection*). Select the best model as $\widehat{\mathcal{M}}^* = \arg \min_{\mathcal{M} \in \mathbb{M}^*} \text{RIC}(\mathcal{M})$.

Example 1. This is an example borrowed from Fan and Lv (2008). Specifically, we fix $d_0 = 1$, $p = 5000$, and $n = 150$. Z_i is generated from $N(0, 1)$. X_i is then simulated as (2.2), where $b_{jk} = 1$ and \tilde{X}_i follows a p -dimensional standard normal distribution. Following Fan and Lv (2008), we assume the first $|\mathcal{M}_T| = 3$ predictors to be relevant and their coefficients are given by $\theta_{0j} = 5$ for $1 \leq j \leq |\mathcal{M}_T|$. Accordingly, $\theta_{0j} = 0$ for every $j > |\mathcal{M}_T|$. Subsequently, Y_i is given by (2.1), where ε_i follows (2.3) with $\alpha_0 = 0.8\sigma_\varepsilon$ and $\tilde{\sigma}_\varepsilon = 0.6\sigma_\varepsilon$. Lastly, σ_ε^2 is particularly selected so that the signal-to-noise ratio, i.e., $\text{SNR} = \text{var}(X_i^\top \theta_0) / \sigma_\varepsilon^2$, is given by 1, 2, or 5.

Signal Noise Ratio	Variable Selection Method	% of Correct Zeros	% of Incorrect Zeros	% of Correct fit	Average Model Size	Absolute Estimation Error
EXAMPLE 1						
1	SIS	100.0	77.2	0.0	1.0	25.4
	PIS	100.0	95.8	0.5	0.1	14.6
	PSS	100.0	95.8	0.5	0.1	14.6
2	SIS	100.0	70.3	0.0	1.0	21.3
	PIS	100.0	46.3	40.0	1.6	7.9
	PSS	100.0	43.3	45.5	1.7	7.4
5	SIS	100.0	67.0	0.0	1.0	18.4
	PIS	100.0	0.2	99.5	3.0	1.0
	PSS	100.0	0.0	100.0	3.0	0.9

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As our first step, we need to estimate the dimension of the latent factor. We find that the first eigenvalue of the matrix $\mathbb{X}\mathbb{X}^\top/(np)$ is as large as $\hat{\lambda}_1 = 35.4\%$ while the second one is as small as $\hat{\lambda}_2 = 3.5\%$. The big difference as demonstrated between $\hat{\lambda}_1$ and $\hat{\lambda}_2$ suggests that the true factor dimension might be $d_0 = 1$. Such a conjecture is formally confirmed by MERC. We then fix $d = 1$ throughout the rest of this example. Thereafter, the factor subspace $\mathcal{S}(\hat{\mathbb{Z}})$ can be estimated and the profiled data $(\hat{\mathbb{Y}}, \hat{\mathbb{X}})$ can be produced.

For a real problem like this, the value of θ_0 is unknown. We thus have to rely on out-of-sample testing to compare different methods' estimation and/or prediction accuracy. We then conducted a total of 200 random experiments. For each experiment, we randomly split the entire dataset $\mathcal{D} = \{1, \dots, 464\}$ into two parts. That is $\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_1$ with $|\mathcal{D}_0| = n_0 = 400$ as the training data and $|\mathcal{D}_1| = n_1 = 64$ as the testing data. Accordingly, we write $\mathbb{X}_0 = \{X_i : i \in \mathcal{D}_0\} \in \mathbb{R}^{n_0 \times p}$, $\mathbb{Y}_0 = \{Y_i : i \in \mathcal{D}_0\} \in \mathbb{R}^{n_0}$, $\mathbb{X}_1 = \{X_i : i \in \mathcal{D}_1\} \in \mathbb{R}^{n_1 \times p}$, and $\mathbb{Y}_1 = \{Y_i : i \in \mathcal{D}_1\} \in \mathbb{R}^{n_1}$. Notations for $(\hat{\mathbb{X}}_0, \hat{\mathbb{X}}_1)$, $(\hat{\mathbb{Y}}_0, \hat{\mathbb{Y}}_1)$, and $(\hat{\mathbb{Z}}_0, \hat{\mathbb{Z}}_1)$ are defined accordingly.

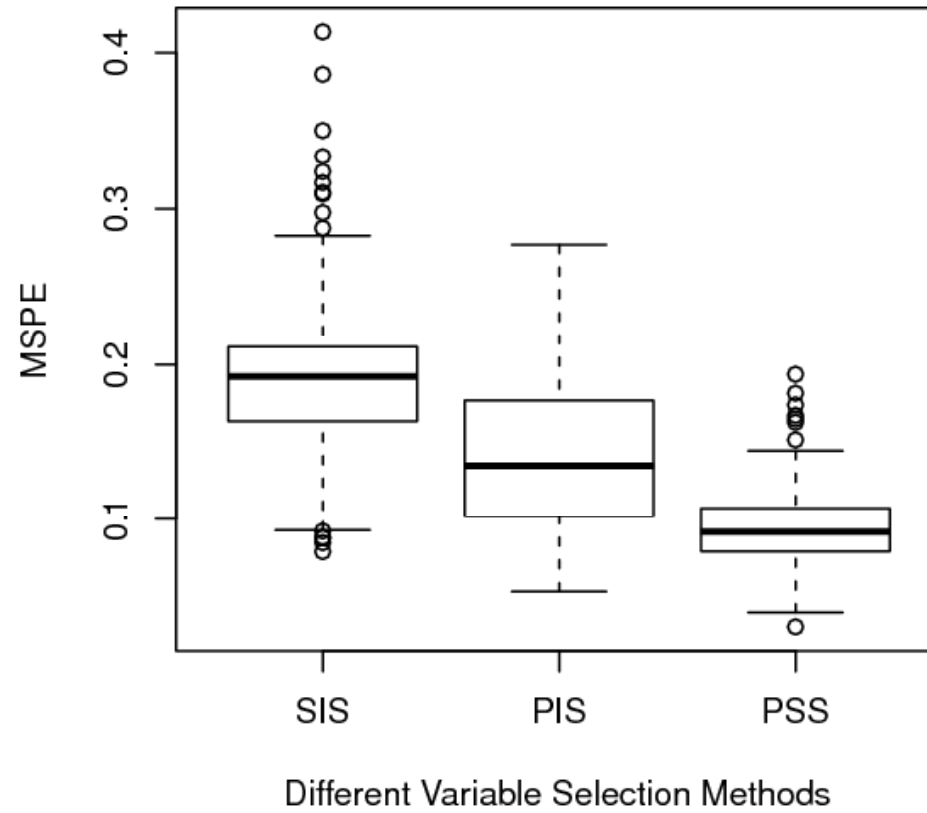


Figure 1: The real supermarket example. Boxplots for the median squared prediction errors (MSPE) based on 200 random replications.

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